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A VISCOELASTIC ANALYSIS OF GROUND-MOVEMENT DUE TO AN ADVANCING COAL FACE

by

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SUMMARY

While and elastic analysis of subsidence problems gives results in accord with practical data for essentially static situations, it fails when applied to the problem of a moving coal face. This failure appears to be due to a time-dependent phenomenon associated with the material in which the excavation is moving. This time effect may be thought of as a lag (that is, the ground does not immediately realize the full extent of the mining disturbance). It is assumed therefore that the ground behaves as though it were a viscoelastic material, exhibiting both initial and delayed elastic states, and it is shown that by a suitable choice of parameters the main features of the practical results can be reproduced. It is also shown that for the model used here the difficulties are due to the large number of parameters involved rather than to any theoretical consideration.

1. *Introduction*

The problem discussed here is to understand and possibly predict the type of subsidence profile that is associated with a steadily moving coal face. The physical situation is shown diagrammatically in Figure I.

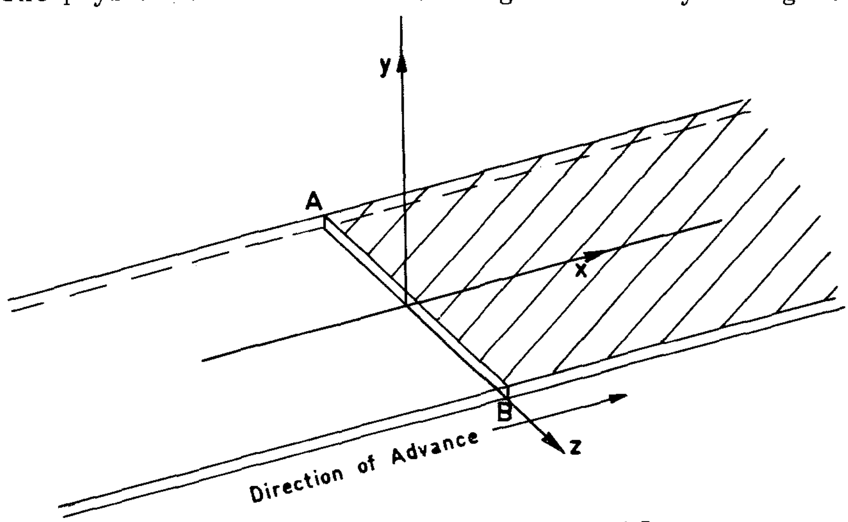

Fig. 1. A B Represents The Moving Coal Face.

Any analysis of ground-movement due to mining is extremely complicated because of the large number of factors involved. It is therefore necessary to make certain simplifying assumptions. These are of two main types: firstly those concerned with the physical properties of the ground and secondly those concerned with the size and shape of the mining disturbance. These may be termed physical and geometrical assumptions respectively.

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(1) PHYSICAL ASSUMPTIONS

The results obtained by assuming the ground to behave like a transversely isotropic elastic medium can be brought into close agreement with experimental data (Berry and Sales 1961 and 1963), when applied to problems which are essentially static, but fail to account for the lag found in the advancing coal face problem (see Figure 2).

When the elastic results are used, it must therefore be assumed that the ground has settled and that time dependent non-elastic effects are no longer operative.

The simplest way of extending the elastic results to bring in time effects is to assume that the ground behaves like a linear viscoelastic solid which exhibits a final equilibrium state. This assumption forms the basis of this paper.

(2) GEOMETRICAL ASSUMPTIONS

It is assumed that the coal seam, and hence the moving disturbance, is horizontal and that the excavation stretches an infinite distance behind the face. That is, it is assumed that the only disturbance affecting the ground-movement is the advancing face. Physically there are four regions at seam level. These are (1) the undisturbed seam, (2) a supported working area, (3) an area which is no longer supported but where the rock roof is self-supporting, and (4) the closed region where the roof has collapsed. Regions (2) and (3) are termed the zone of incomplete convergence or simply the non-closed region. For a deep mine (of the order of 2,000 feet), it is found that the zone of incomplete convergence is considerably less than a tenth of the mine depth, denoted by h.

In this paper the excavation is considered closed. (That is the non-closed region is taken to be small compared with the depth of the excavation). This assumption simplifies the mathematical boundary problem. Also from the elastic analysis of Berry and Sales it would appear that, at least as far as the surface movement (subsidence) is concerned, the degree of closure does not affect the result to any great extent. Further in discussing the experimental data a rough allowance is made for the non-closed region, by moving the origin of the elastic curve back a distance equal to a tenth of the mine depth, which is a distance greater than the non-closed region.

In order for the problem to be treated as two dimensional (i.e. as a plane-strain problem) it is necessary to assume that the seam is of an infinite width. This would be reasonable if the ratio of seam width to mine depth was large; the reason for the assumption is, however, mathematical rather than geometrical.

The final assumption is that a steady state has been attained such that with respect to axes fixed on the disturbance the stress-strain pattern is stationary. This will be true if the excavation has been moving with constant velocity for *some* time, which in this context is of the order of time taken for the face to advance a distance comparable with the depth of the excavation.

It is convenient at this point to define the coordinate axes used in the mathematical problem. As shown in figure 1, the x-axis is taken along the line of advance of the seam, the z-axis is taken in the seam at right angles to the x-axis, and the y-axis is the upward vertical. The origin at time $t = o$, is at the centre of the face, which is moving uniformly with velocity c, in the x-direction.

(3) REASONS FOR FAILURE OF THE ELASTIC RESULT

The subsidence profile for the elastic problem can be expressed as

$$S = \frac{V}{|V_m|} = \frac{1}{\pi}\left\{ \frac{\alpha_1}{\alpha_1 - \alpha_2} \tan^{-1} x\alpha_1 - \frac{\alpha_2}{\alpha_1 - \alpha_2} \tan^{-1} x\alpha_2 \right\} - \tfrac{1}{2}, \qquad (1)$$

where V is the subsidence, V_m is the maximum value of V, α_1 and α_2 are the roots of a certain fourth order equation and have positive real parts (N. B. as the α's are complex conjugates (1) will always yield a real value for S), and the unit in the x direction is the depth of the excavation. This result can be obtained from the work of Berry and Sales by a limiting procedure but it is obtained here from the viscoelastic analysis. The elastic curve has two main properties; first it is skewsymmetrical about $S = -\frac{1}{2}$ and secondly this value occurs at x = 0. The average of eleven excavations in the Yorkshire coal field has been presented by Wardell and it is found that the curve is not skew-symmetrical about $S = -\frac{1}{2}$, and $S = -\frac{1}{2}$ occurs around x = -0.25 (i.e. at one quarter of the depth of the excavation behind the face). Here this later result is called the "lag". The above two features are characteristics of most subsidence profiles of this type (deep exacavattions) and are in direct contradiction to the elastic result.

It may be argued that the "lag" is due to non-closure effects but even when the elastic curve is set back (translated) a tenth of the mine depth (a figure in excess of the length on non closure found in practice), there is still a considerable lag.

In Figure 2 the values $\alpha_1 = 4.45$ and $\alpha_2 = 0.45$ are taken in accord with the work of Berry (1963).

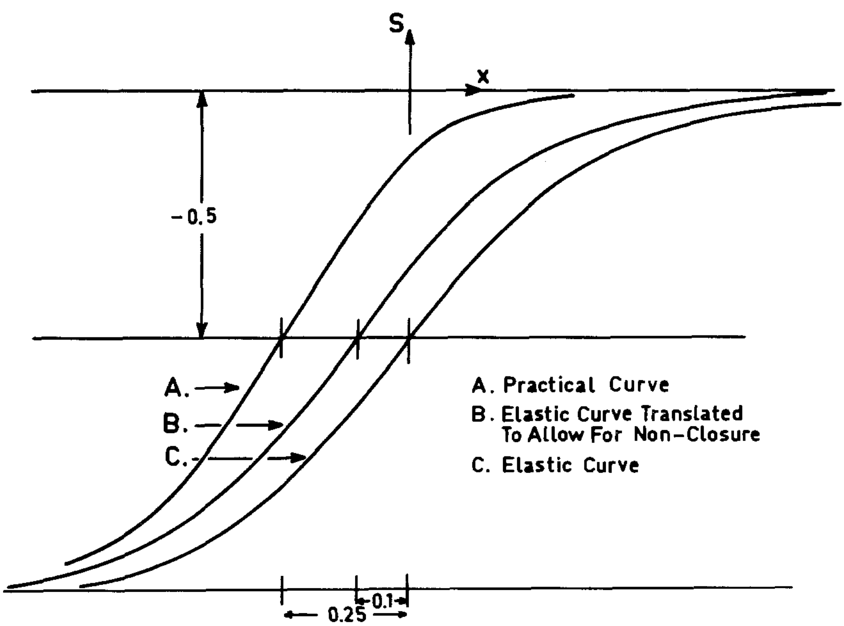

Fig. 2. Graph Of Subsidence, S, Against $X = \frac{x}{h}$.

2. *Mathematical Formulation*

The mathematical problem treated in this paper is one of plane-strain, but in order to relate the final viscoelastic state with meaningful physical constants, it is necessary to state the elastic stress-strain relations in their full three dimensional form.

The stress-strain relations for a transversely isotropic elastic medium with the y-axis as the axis of symmetry can be written as

$$E_x\epsilon_x = E_x \frac{\partial u}{\partial x} = \sigma_x - \nu_y \sigma_y - \nu_x \sigma_z$$

$$E_x\epsilon_y = E_x \frac{\partial v}{\partial y} = - \nu_y \sigma_x + \left(\frac{E_x}{E_y}\right)\sigma_y - \nu_y \sigma_z$$

$$E_x\epsilon_z = E_x \frac{\partial w}{\partial z} = - \nu_x \sigma_x - \nu_y \sigma_y + \sigma_z$$

$$2M \epsilon_{xy} = M \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}$$

$$2M \epsilon_{xz} = M \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \tau_{xz}$$

$$2E_x\epsilon_{xz} = E_x \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = (1 + \nu_x)\, \tau_{xz}$$

where E_x and E_y are Young's moduli, ν_x and ν_y are Poisson's ratios and M is a shear modulus (see Berry and Sales 1961). N. B. The symbols E_x, E_y, ν_x, ν_y and M are physical elastic constants, however when dealing with the viscoelastic model they will be treated, essentially as time-dependent linear operators.

For the case of plane strain in the (x, y) plane, $\epsilon_z = \epsilon_{yz} = \epsilon_{xz} = 0$, and the stress-strain relationship reduces to

$$E_x\epsilon_x = E_x \frac{\partial u}{\partial x} = (1 - \nu_x^2)\, \sigma_x - \nu_y (1 + \nu_x)\, \sigma_y,$$

$$E_x\epsilon_y = E_x \frac{\partial v}{\partial y} = - \nu_y (1 + \nu_x)\, \sigma_x + \left(\frac{E_x}{E_y} - \nu_y^2\right)\sigma_y,$$

$$M \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy},$$

and $\qquad \sigma_z = \nu_x \sigma_x + \nu_y \sigma_y.$

These equations can conveniently be written

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= s_1 \sigma_x + s_2 \sigma_y \\ \frac{\partial v}{\partial y} &= s_2 \sigma_x + s_3 \sigma_y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 2\, s_4\, \tau_{xy} \end{aligned}\right\}, \qquad (2)$$

with $\qquad s_1 = (1 - \nu_x^2)/E_x, \; s_2 = - \nu_y(1 + \nu_x)/E_x,$

$\qquad s_3 = 1/E_y - \nu_y^2/E_x,$ and $s_4 = 1/2M.$

For isotropy $\qquad \nu_x = \nu_y = \nu, \; E_x = E_y = E,$

and $\quad 1/2M = (1 + \nu)/E$; that is $s_1 = s_3$

and $\quad s_4 = s_1 - s_2$.

Equations of the same form as (2) can be used to characterize a transversely isotropic, linear, viscoelastic medium provided, s_1, s_2, s_3, and s_4 are now interpreted as time dependent linear operators. By means of Boltzman's superposition principle the general form of these operators is given by hereditary integrals as used by Volterra (1930).

The form of stress-strain relationship used in this paper will therefore be taken as

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \int_{-\infty}^{t} s_1(t - t_1)\frac{\partial\sigma_x}{\partial t_1}dt_1 + \int_{-\infty}^{t} s_2(t - t_1)\frac{\partial\sigma_y}{\partial t_1}dt_1 , \\
\epsilon_y &= \frac{\partial v}{\partial y} = \int_{-\infty}^{t} s_2(t - t_1)\frac{\partial\sigma_x}{\partial t_1}dt_1 + \int_{-\infty}^{t} s_3(t - t_1)\frac{\partial\sigma_y}{\partial t_1}dt_1 , \\
2\epsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\int_{-\infty}^{t} s_4(t - t_1)\frac{\partial\tau_{xy}}{\partial t_1}dt_1 ,
\end{aligned} \tag{3}$$

where t is the present time and the s_j's are creep functions which are functions of $(t-t_1)$, assumed to be present, - that is they are memory functions, so that the material has some integrated recollection of earlier states.

In addition to (3) we also have the equations of motion, which, for the problems considered here, are assumed to be replaceable by the "quasi-static" equations

$$\left.\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \\
\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} &= 0
\end{aligned}\right\} \tag{4}$$

Consider now the last equation of (3)

$$\epsilon_{xy} = \int_{-\infty}^{t} s_4(t - t_1)\frac{\partial\tau_{xy}}{\partial t_1}\,dt,.$$

With the assumption of a steady state this equation can be replaced by

$$\epsilon_{xy} = \int_{-\infty}^{\tau} s_4(\tau - \tau_1)\frac{\partial\tau_{xy}}{\partial\tau_1}\,d\tau_1 ,$$

where $\quad \tau = t - \frac{x}{c}$ and $\tau_1 = t_1 - \frac{x}{c}$.

(Mathematically this implies that the face has been moving uniformly from $t = -\infty$ to the present time, physically however it is only necessary for the face to have been moving for a sufficiently long time that the start of mining operation is of considerably less importance to the subsidence profile than the moving face - see introduction).

This may be integrated by parts to give

$$\epsilon_{xy} = \int_{-\infty}^{\tau} s_4' (\tau - \tau_1) \tau_{xy} \, d\tau_1 + s_4(o) \; \tau_{xy} , \tag{5}$$

provided $\tau_{xy}(-\infty, y) = 0$ that is τ_{xy} is zero at an infinite distance ahead of the face, irrespective of the value of y, (the prime denotes a differential coefficient).

Before proceeding further it is necessary to define the Fourier transform of a function. The definition used in this paper is as follows: if f is a function of τ then $\bar{f}$ the transform of f with respect to τ is given by

$$\bar{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) \, e^{i\tau\omega} d\tau,$$

the bar being used to denote transformed quantities.

Taking the transform of (5) with respect to τ and making use of the convolution theorem gives

$$\bar{\epsilon}_{xy} = \hat{s}_4 \; \bar{\tau}_{xy} \tag{6}$$

where $\qquad \hat{s}_4 = \left(\sqrt{2\pi} \; \bar{s}_4' (\omega) + s_4(o)\right) .$

It is clear that the other integrals occurring in (3) may be treated similarly and upon noting $\frac{\partial f}{\partial x} = - \frac{1}{c} \frac{\partial f}{\partial \tau}$, by the steady state assumption, equation (3) and (4) can be replaced by

$$\left.\begin{aligned}
\frac{i\omega}{c} \bar{u} &= \hat{s}_1 \; \bar{\sigma}_x + \hat{s}_2 \; \bar{\sigma}_y , \\
\frac{d\bar{v}}{dy} &= \hat{s}_2 \; \bar{\sigma}_x + \hat{s}_3 \; \bar{\sigma}_y , \\
\frac{d\bar{u}}{dy} + \frac{i\omega}{c} \bar{v} &= 2 \; \hat{s}_4 \; \bar{\tau}_{xy} , \\
\frac{i\omega}{c} \bar{\sigma}_x + \frac{d\bar{\tau}_{xy}}{dy} &= 0 , \\
\frac{i\omega}{c} \bar{\tau}_{xy} + \frac{d\bar{\sigma}_y}{dy} &= 0 .
\end{aligned}\right\} \tag{7}$$

From (7) it is easily deduced that

$$\hat{s}_1 \frac{d^4 \bar{f}}{dy^4} - 2 \left(\frac{\omega}{c}\right)^2 (\hat{s}_2 + \hat{s}_4) \frac{d^2 \bar{f}}{dy^2} + \left(\frac{\omega}{c}\right)^4 \hat{s}_3 \; \bar{f} = 0, \tag{8}$$

where $\bar{f}$ is any one of $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$, $\bar{u}$ or $\bar{v}$ (c.f. Berry and Sneddon 1958).

The general solution of (8) is

$$\bar{f} = A \; e^{-\frac{\omega y}{c\alpha_1}} + B \; e^{\frac{-\omega y}{c\alpha_2}} + C \; e^{\frac{+\omega y}{c\alpha_1}} + D \; e^{\frac{+\omega y}{c\alpha_2}},$$

with $\pm\,\alpha_1$ and $\pm\,\alpha_2$ the roots of

$$\hat{s}_3\,\alpha^4 - 2\,(\hat{s}_2 + \hat{s}_4)\alpha^2 + \hat{s}_1 = 0. \tag{9}$$

The boundary conditions now serve to determine A, B, C and D.

3. *Infinite Medium*

The physical assumption of complete closure is taken to correspond to a discontinuity in displacement (dislocation) of amount $-\ell/2$ and this leads to the following boundary condition, on $y = 0$,

$$\begin{array}{ll} \Delta v = -\ell/2 & x < ct \\ \Delta v = 0 & x > ct \\ \tau_{xy} = 0 & \text{for all } x. \end{array}$$

where Δf denotes the discontinuity in f across $y = 0$.

In terms of τ these are, on $y = 0$,

$$\left.\begin{array}{ll} \Delta v = -\ell/2 & \tau > 0 \\ \Delta v = 0 & \tau < 0 \\ \tau_{xy} = 0 & \text{for all } \tau. \end{array}\right\} \tag{10}$$

The boundary value problem is best solved by assuming that $\bar{\sigma}_4 = \bar{P}(\omega)$ is known on $y = 0$ and then using (10) to determine P.

The solution is given by

$$\bar{\sigma}_4 = \frac{\bar{P}}{\alpha_1 - \alpha_2}\left\{\alpha_1\, e^{\frac{-|\omega y|}{c\alpha_1}} - \alpha_2\, e^{\frac{-|\omega y|}{c\alpha_2}}\right\}$$

$$\bar{\sigma}_x = \frac{\bar{P}}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)}\left(\alpha\, e^{\frac{-|\omega y|}{c\alpha_2}} - \alpha_2\, e^{\frac{-|\omega y|}{c\alpha_1}}\right),$$

$$\tau_{xy} = \frac{-c\bar{P}}{i\omega\,(\alpha_1 - \alpha_2)}\left(e^{\frac{-|\omega y|}{c\alpha_1}} - e^{\frac{-|\omega y|}{c\alpha_2}}\right), \tag{11}$$

$$\bar{u} = \frac{c\bar{P}}{i\omega(\alpha_1 - \alpha_2)}\left\{\left(\alpha_1\hat{s}_2 - \frac{\hat{s}_1}{\alpha_1}\right) e^{\frac{-|\omega y|}{c\alpha_1}} - \left(\alpha_2\hat{s}_2 - \frac{\hat{s}_1}{\alpha_2}\right) e^{\frac{-|\omega y|}{c\alpha_2}}\right\}$$

$$\bar{v} = \frac{-c\bar{P}}{|\omega|\,(\alpha_1 - \alpha_2)}\left\{(\hat{s}_3\alpha_1^2 - \hat{s}_2)\, e^{\frac{-|\omega y|}{c\alpha_1}} - (\hat{s}_3\alpha_2^2 - \hat{s}_2)\, e^{\frac{-|\omega y|}{c\alpha_2}}\right\}$$

and $$\bar{P} = \frac{\ell i\omega}{|\omega|c\;\hat{s}_3(\alpha_1+\alpha_2)2\sqrt{2\pi}}$$

It should be noted that (11) satisfies the conditions $\bar{\sigma}_y = \bar{p}(\omega)$, $\tau_{xy} = 0$, on $y = 0$, and thus the above solution can be used with appropriate functions $\bar{p}(\omega)$, for the problem of arbitrary σ_y on a shear free surface $y = 0$.
The above solution is essentially that for a dislocation moving steadily through an infinite transversely isotropic, viscoelastic medium.

For the case of isotropic ($\alpha_1 = \alpha_2$) and for a creep function s_3 of the standard linear solid type ($s_3 = s\ (1 - r\ e^{-\delta t})$) the equation for $\bar{P}$ can be inverted in terms of exponential integrals and it agrees in form with a similar problem solved by Eshelby (1949) and (1961).

4. *Half plane*

In section 3. the solution represented a discontinuity in an infinite medium located on $y = 0$. The present section gives the solution representing the same discontinuity located on $y = -h$, with a traction free boundary at $y = 0$.
Firstly move the discontinuity in section 3. to $y = -h$ and denote the stresses and displacements by the subscript $-h$. Secondly move the discontinuity in section 3. to $y = +h$ and denote the stresses and displacements by the subscripts $+h$. Then since all the equations are linear the sum of these two will also satisfy the basic equations, and it will have the required discontinuity of displacement on $y = -h$. It is found that on $y = 0$ $\tau_{xy} = 0$ but $\sigma_y = P'$ (say) where $\bar{P}' = 2\bar{P}\left(\dfrac{\alpha_1 e^{-\frac{\omega h}{c\alpha_1}} - \alpha_2 e^{-\frac{\omega h}{c\alpha_2}}}{\alpha_1 - \alpha_2}\right)$. In order to solve the problem therefore it is necessary to add a further stress field which satisfies the conditions $\sigma_y = -P'$ and $\tau_{xy} = 0$, on $y = 0$, and there are no displacement discontinuities introduced by this field in the lower half plane. This third field is therefore given by (11) with $\bar{P}$ replaced by $-\bar{P}'$. Let the stresses and displacements associated with this problem be denoted by primes.
The complete solution can be represented symbolically therefore by

$$\sigma_y = \sigma_{y_h} + \sigma_{y_{-h}} + \sigma_4'$$

$$\sigma_x = \sigma_{x_h} + \sigma_{x_{-h}} + \sigma_x'$$

$$\tau_{xy} = \tau_{xy_h} + \tau_{xy_{-h}} + \tau_{xy}'$$

$$u = u_h + u_{-h} + u'$$

and $$v = v_h + v_{-h} + v'.$$

As the practical data is concerned with surface movement (subsidence) the rest of the paper will be concerned with the value of v (the vertical movement) on $y = 0$.
It is clear from the above that on $y = 0$, $v = v'$ so $\bar{v} = \bar{v}'$ which from (11) is given by

$$\bar{v}(o;\omega) = \bar{v}'(o,\omega) = \bar{V} = \bar{V}' = \frac{c\bar{P}'}{\omega}\ \hat{s}_3(\alpha_1 + \alpha_2),$$

when the value of $\bar{P}'$ is substituted this becomes

$$\overline{V} = \overline{V}' = \frac{2c\overline{P}}{\omega}\hat{s}_3(\alpha_1 + \alpha_2)\left\{\frac{\alpha_1 e^{\frac{-|\omega|h}{c\alpha_1}} - \alpha_2 e^{\frac{-|\omega|h}{c\alpha_2}}}{\alpha_1 - \alpha_2}\right\},$$

which again from (11) is

$$\overline{V} = \frac{\ell i}{\sqrt{2\pi}\omega}\left(\frac{\alpha_1 e^{\frac{-\omega h}{c\alpha_1}} - \alpha_2 e^{\frac{-\omega h}{c\alpha_2}}}{\alpha_1 - \alpha_2}\right) \tag{12}$$

For the elastic problem with the α's real (12) can be inverted to give (see Erdelyi 1954)

$$S = \frac{V}{\ell} = \frac{1}{\pi}\left\{\frac{\alpha_1}{\alpha_1 - \alpha_2}\tan^{-1}X\alpha_1 - \frac{\alpha_2}{\alpha_1 - \alpha_2}\tan^{-1}X\alpha\right\}, \tag{13}$$

where $X = \dfrac{x - ct}{h}$.

In (12) as $X \to -\infty$, $S \to -\frac{1}{2}$ and as $X \to +\infty$ $S \to \frac{1}{2}$ but physically as $X \to -\infty$, $S \to -1$, and as $X \to +\infty$, $S \to 0$; this can be remedied by substracting a half from each side, this is justified as this is a rigid body displacement.

Hence for the elastic curve.

$$S = \frac{V}{\ell} = \frac{1}{\pi}\left\{\frac{\alpha_1}{\alpha_1 - \alpha_2}\tan^{-1}X\alpha_1 - \frac{\alpha_2}{\alpha_1 - \alpha_2}\tan^{-1}X\alpha_1\right\} - \frac{1}{2}, \tag{14}$$

where in accord with the work of Berry (1963) the α's are taken as real, for complex α's the result contains logarithmic terms. From (12) it is found that

$$S = \frac{V}{\ell} = \frac{1}{\pi}\int_0^\infty \frac{A(\omega)}{\omega}\sin\frac{Xh}{c}\omega\, d\omega + \frac{1}{\pi}\int_0^\infty \frac{B(\omega)}{\omega}\cos\frac{Xh\omega}{c}d\omega - \frac{1}{2}, \tag{15}$$

where $A(\omega)$ and $-B(\omega)$ are the real and imaginary parts of

$$\frac{\alpha_1 e^{\frac{-h\omega}{c\alpha_1}} - \alpha_2 e^{\frac{-h\omega}{c\alpha_2}}}{\alpha_1 - \alpha_2}$$

and the $\frac{1}{2}$ has been introduced so that (15) reduces to (14) when the α's are real and constant

The integrals $\int_0^\infty A(\omega)\,d\omega$ and $\int_0^\infty \frac{B(\omega)}{\omega}\,d\omega$, can be given physical interpretations as follows.

In (15) put $X = 0$ then

$$S(o,o) = \frac{v}{\ell}(o,o) = \frac{1}{\pi}\int_0^\infty \frac{B(\omega)}{\omega}\,d\omega - \frac{1}{2},$$

and thus $\frac{1}{\pi}\int_0^\infty \frac{B(\omega)}{\omega}\,d\omega$, measures the deviation of $S(o,o)$ from half subsidence, this integral will be called the "lag integral".

If (15) is differentiated with respect to X and then X is put equal to zero, it is found that

$$\left(\frac{ds}{dX}\right)_o = \frac{h}{c\pi}\int_0^\infty A(\omega)d\omega$$

So $\int_0^\infty A(\omega)d\omega$, is a measure of the slope of the subsidence curve at the origin.

These two parameters are discussed further in the next section where some numerical results are presented.

5. *Numerical Analysis*

From (15) it is clear that the subsidence curve depends on the $\hat{s}$'s by way of α_1 and α_2; also from equation (9) it can be seen that of the $\hat{s}$'s all behave similarly then α_1 and α_2 are constants and (15) gives a solution similar in form to the elastic result.

The simplest way to ensure that the α's are not constants, but functions of ω is to assume that the ground behaves differently in its responses to shear and simple extension.

It is therefore assumed that s_4 is a constant while for $j \neq 4$, $s_j = s_j(\infty)(1 - r\,e^{-\delta t})$. In a one dimensional problem a creep function of the above type is equivalent to a spring in series with a Voigt model and it is sometimes called a standard linear solid. It exhibits initial and final elastic states together with viscous flow. (Note the analysis is not significantly altered if s_4 is assumed to creep as above and the other s_j's are regarded as constants).

It is found on using (6) that

$$\hat{s}_j(\omega) = s_j(\infty)\left\{\frac{\delta - (1-r)}{\delta - i\omega}\right\} \tag{16}$$

for $j \neq 4$ while $\hat{s}_4 = s_4(\infty)$.

In accord with the work of Berry (1963) it is assumed that in the final elastic state $\alpha_1 = 4.45$ and $\alpha_2 = 0.45$.

This means that

$$s_2(\infty) + s_4(\infty) = 10\, s_3(\infty)$$

and $$s_1(\infty) = 4\, s_3(\infty) \tag{17}$$

From (9) it is seen that

$$\alpha_1^2 + \alpha_2^2 = 2\frac{(\hat{s}_2+\hat{s}_4)}{\hat{s}_3} = 2\,\frac{\left(10\delta - 10i\omega + ri\omega\,\dfrac{s_2(\infty)}{s_3(\infty)}\right)}{\delta - i\omega\,(1-r)}$$

and $$\alpha_1^2\,\alpha_1^2 = \frac{\hat{s}_1}{\hat{s}_3} = \frac{s_1(\infty)}{s_3(\infty)} = 4 \tag{18}$$

i.e. $\alpha_1\alpha_2 = 2.$

But $\dfrac{s_2(\infty)}{s_3(\infty)} = 4\,\dfrac{s_2(\infty)}{s_1(\infty)} = -\dfrac{4\,\nu_y}{1-\nu_x}$, (from (2)), and if it supposed that $o < \dfrac{\nu_x}{\nu_y} < \frac{1}{2}$ then $-4 < \dfrac{s_2(\infty)}{s_3(\infty)} < o$. As there is no experimental data to suggest a value for this ratio and as it does not significantly affect the value of $\alpha_1^2 + \alpha_2^2$ it seems reasonable to take this ratio as -2 that is in the centre of the range.

With the above assumption equation (18) gives

$$\alpha_1^2 + \alpha_2^2 = 4\,\frac{(5\delta - 5i\omega - ri\omega)}{\delta - i\omega\,(1 - r)} = 4K \quad \text{(say)}$$

and $\quad \alpha_1 \alpha_2 = 2$

The above give

$$\begin{aligned} \alpha_1 &= \sqrt{K+1} + \sqrt{K-1} \\ \alpha_2 &= \sqrt{K+1} - \sqrt{K-1} \end{aligned} \qquad (19)$$

The values of $A(\omega)$ and $\frac{B(\omega)}{\omega}$ were found for various values of $\frac{h}{c}$, r and δ. Because of the physical meanings of $\int_0^\infty A(\omega)d\omega$ and $\int_0^\infty \frac{B(\omega)}{\omega}\,d\omega$ it was decided to see if, for a given h/c, values r and δ could be found to give the same sort of lag and slope found in practice (i.e. 0.23 and 1.8 respectively). The results for h/c = 3.5 (h in feet, c in feet per year) are given as these are fairly typical.

Table I.

h/c = 3.5	δ, inverse years	"lag"	"slope"
r = 0.2	0.1	0.005	2.2
	0.4	0.013	2.2
	2.0	0.019	1.9
r = 0.5	0.1	0.02	2.3
	0.4	0.03	2.2.
	2.0	0.04	2.1
r = 0.8	0.1	0.05	3.8
	0.4	0.10	3.3
	2.0	0.20	2.2
r = 1.0	0.1	0.26	22
	0.4	0.32	17
	2.0	0.38	13

From the table it is seen that $\delta > 2$ (inverse years) and that of the tabulated values r = 0.8 gives the most reasonable values for the two integrals. The results are presented graphically in Figures (3) and (4).

6. *Dimensionless Parameters*

If in equation (15) we change the variable of integration from ω to $W = \frac{h\omega}{c}$, (15) becomes

$$S = \frac{V}{\ell} = \frac{1}{\pi}\int_0^\infty A'\,\frac{\sin XW}{W}\,dW + \frac{1}{\pi}\int_0^\infty B'\,\frac{\cos XW}{W}\,dW$$

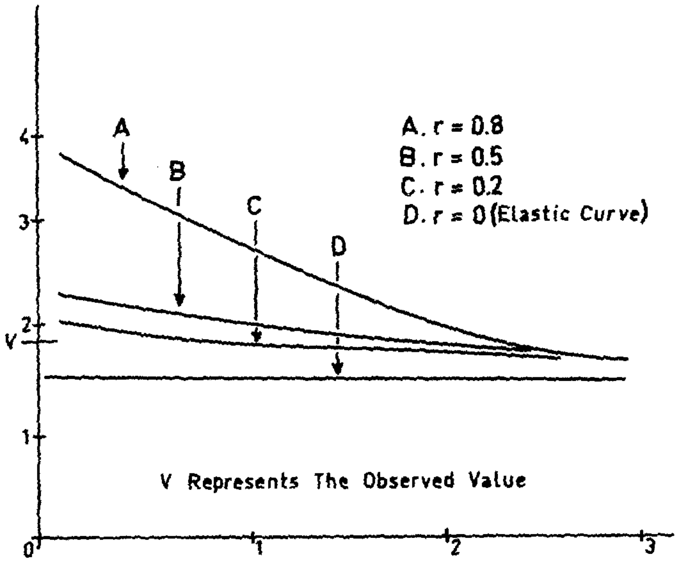

Fig. 3. Graph Of $\frac{h}{c\pi}\int^{\infty} A(\omega)\,d\omega$ Against δ.

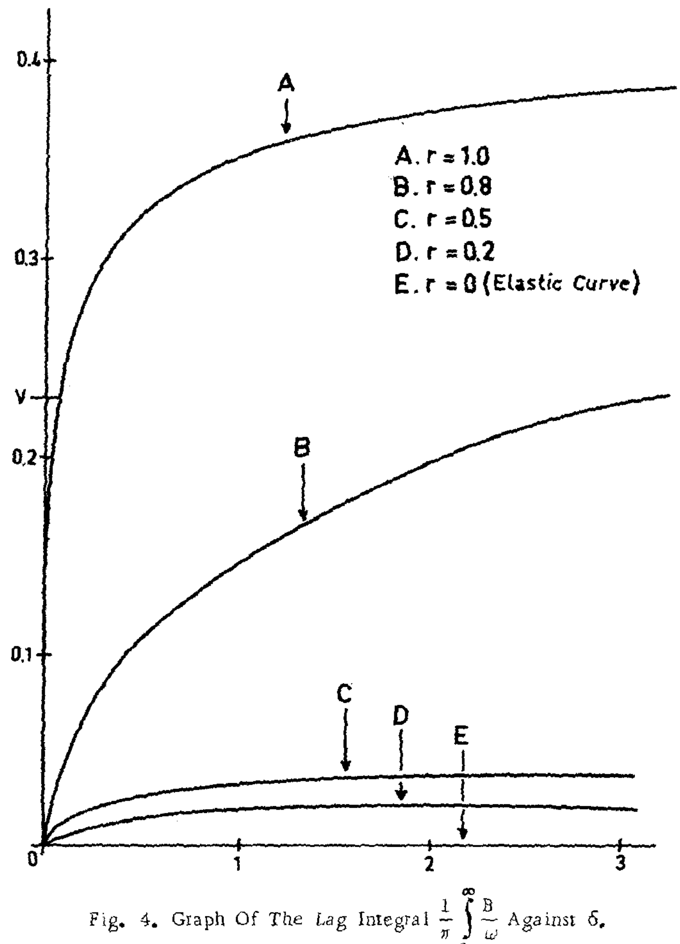

Fig. 4. Graph Of The Lag Integral $\frac{1}{\pi}\int_0^{\infty}\frac{B}{\omega}$ Against δ.
V is The Observed Value.

where $$A - iB = \frac{\alpha_1 e^{\frac{-W}{\alpha_1}} - \alpha_2 e^{\frac{-W}{\alpha_2}}}{\alpha_1 - \alpha_2}, \quad \text{with } \alpha_1$$

and α_2 now expressed in terms of W. Making the same physical assumptions as in section 5. α_1 and α_2 are still given by (19) but now

$$K = \frac{5M - 5iW - riW}{M - iW\,(1-r)}$$

where $M = \frac{\delta h}{c}$.

From this the value of S is seen to depend on two dimensionless parameters r and M. The parameter r is a measure of the ratio of the initial values of the creep functions to their final values. If $r = 0$ ($K=5$) then the result reduces to the elastic problem.

$M = \frac{\delta h}{c}$ is the ratio of two times: δ^{-1} is a "natural time" due to the assumed viscoelastic properties of the ground, while h/c is an imposed time due to the mining conditions (it is in fact the time taken for the face to advance a distance equal to the depth of the excavation).

When M is small (a fast excavation) then $K = \frac{5 + r}{1 - r}$, this is the form of the elastic result with the creep functions having their initial values.

When M is large (a slow excavation) then $K = 5$, this is the form of the elastic result with the creep functions having their final values.

7. *Conclusions*

At present insufficient practical and theoretical investigation of time effects in mining preclude any firm conclusion on the feasibility or usefulness of a viscoelastic analysis. However, this paper shows that with an appropriate choice of δ and r the main features of the subsidence profile can be reproduced; further the term involving $B(\omega)$ in equation (15) will always procedure a lag and this term also destroys the symmetry found in the elastic case. It would seem therefore that for the moving coal face the model used here should prove more suitable than a purely elastic one.

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